VECTORS

summary:
1. Introduction
2. scalar / Dot product
3. Vector / cross product
4. Equations of lines
5. Equations of planes

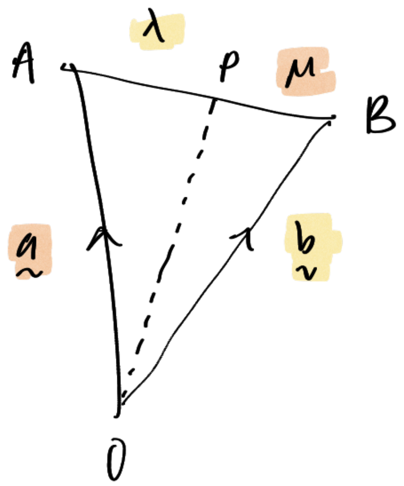
1. INTRO

if $\underset{\sim}{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $|\underset{\sim}{a}| = \sqrt{x^2+y^2+z^2}$

magnitude / length of $\underset{\sim}{a}$

unit vector of $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

★ RATIO THEOREM



$$\overrightarrow{OP} = \frac{\mu \underset{\sim}{a} + \lambda \underset{\sim}{b}}{\mu + \lambda}$$

2. SCALAR / DOT PRODUCT

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

you get a number/scalar, hence the name

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

PROPERTIES:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$$

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$$

$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{c} + \underset{\sim}{d}) = \underset{\sim}{a} \cdot \underset{\sim}{d} + \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{d}$$

$$\lambda \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot (\lambda \underset{\sim}{b}) = \lambda (\underset{\sim}{a} \cdot \underset{\sim}{b})$$

☆ $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$ OR $|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

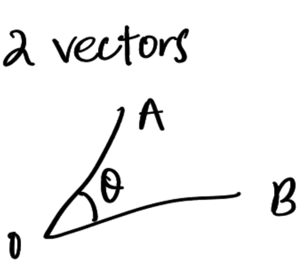
APPLICATIONS:

i. when **perpendicular**, ALSO THINK OF DOT PRODUCT!

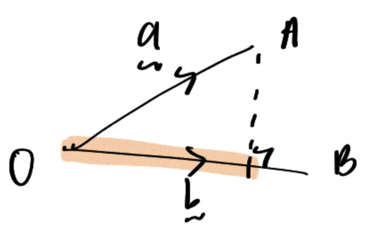
$\underset{\sim}{a} \perp \underset{\sim}{b}$ only when $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

ii. finding **angle** between 2 vectors

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$



iii. **length of projection**



of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

$$= |\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}| = \frac{1}{|\underset{\sim}{b}|} |\underset{\sim}{a} \cdot \underset{\sim}{b}|$$

3. VECTOR PRODUCT

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

a vector

properties:

$$\underset{\sim}{a} \times \underset{\sim}{b} = -(\underset{\sim}{b} \times \underset{\sim}{a})$$

$$\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$$

$$(\underset{\sim}{a} + \underset{\sim}{b}) \times (\underset{\sim}{c} + \underset{\sim}{d}) = \underset{\sim}{a} \times \underset{\sim}{d} + \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{d}$$

$$\lambda \underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{a}(\lambda \underset{\sim}{b}) = \lambda(\underset{\sim}{a} \times \underset{\sim}{b})$$

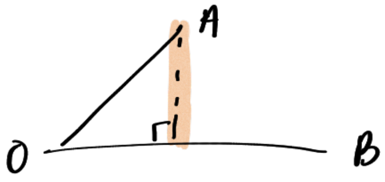
$$\star \quad \underset{\sim}{a} \times \underset{\sim}{a} = 0$$

APPLICATIONS:

i. when parallel, think about cross product

$\underset{\sim}{a} \parallel \underset{\sim}{b}$ only when $\underset{\sim}{a} \times \underset{\sim}{b} = 0$

ii. perpendicular / shortest distance

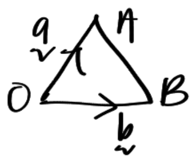


$$= |\underset{\sim}{a} \times \hat{\underset{\sim}{b}}| = \frac{1}{|\underset{\sim}{b}|}|\underset{\sim}{a} \times \underset{\sim}{b}|$$

$$= \frac{1}{|\underset{\sim}{b}|}\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right|$$

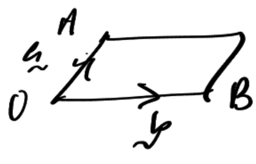
$$= \frac{1}{|\underset{\sim}{b}|}\sqrt{x^2+y^2+z^2}$$

iii. Area of $\triangle OAB$



$$= \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$$

iv. Area of ▱ parallelogram



$$= |\underset{\sim}{a} \times \underset{\sim}{b}|$$

4. EQUATION of STRAIGHT LINES

→ A. Forms
→ B. Point & line
→ C. Line & Line

A. Forms of eqns.

i. Vector form: $l: \underset{\sim}{r} = (\quad) + \lambda(\quad)$

↑ position vector ↑ direction vector $\underset{\sim}{d}$

ii. Parametric

$$x = a_1 + \lambda b_1$$
$$y = a_2 + \lambda b_2$$
$$z = a_3 + \lambda b_3$$

convert to vector: $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

iii. cartesian

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

convert to vector: $\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

B. POINT & LINE

i. determining if a point lies on line

you have $l: \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ and point $\underset{\sim}{c}$

just sub $\underset{\sim}{c}$ into l. so like

① $\underset{\sim}{c} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

② then solve for λ

③ sub λ into l and you get c

ii. Finding foot of perpendicular from point to a line

① let F be the foot of the perpendicular from C to l

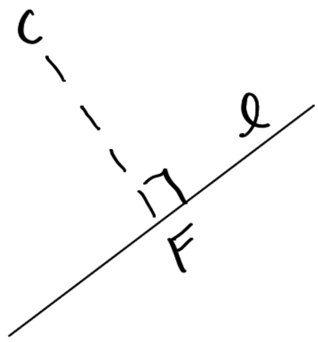
② F lies on l, so $\vec{OF} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$, for some $\lambda \in \mathbb{R}$

③ $\vec{CF} = \vec{OF} - \vec{OC}$

④ since $\vec{CF} \perp l$, $\vec{CF} \cdot (\underset{\sim}{b}) = 0$

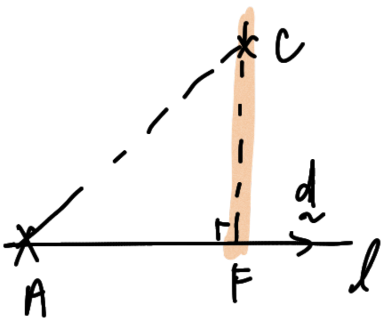
solve for λ

⑤ sub λ into $\vec{OF} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ = ANS



iii. Finding shortest distance from pt. to line.

[when Qn doesnt ask for point, use this method!]



① Find $\vec{AC} = \vec{OC} - \vec{OA}$ (C is given in Qn. A is usually position vector of l or ANY point on the line)

② use vector product

shortest distance

$= |\vec{AC} \times \hat{\underset{\sim}{b}}|$

C: LINE X LINE

3 SCENARIOS:

1. intersecting
2. parallel (if $\underset{\sim}{d}_1 = k\underset{\sim}{d}_2$)
3. skew (not intersecting, not parallel)

i. determining if 2 lines intersect

① Equate 2 lines together

$$\underset{\sim}{a} + \lambda \underset{\sim}{b} = \underset{\sim}{c} + \mu \underset{\sim}{d}$$

② Form 3 equations involving λ and μ

Eg: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$

then the 3 eqns are:

$1 + 4\lambda = 7 + 10\mu$ — ①

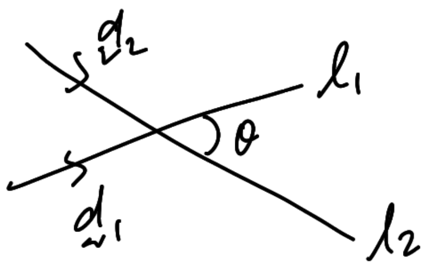
$2 + 5\lambda = 8 + 11\mu$ — ②

$3 + 6\lambda = 9 + 12\mu$ — ③

③ check if there is a unique solution for λ and μ.

if yes, then the lines intersect. ☺

ii. Acute Angle bet. 2 lines



$$\cos\theta = \frac{|\underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2|}{|\underset{\sim}{d}_1||\underset{\sim}{d}_2|}$$

5. Eqns of planes

- A: forms of equations
- B: point & plane
- C: line & plane
- D: plane & plane

A: forms of equations.

i. vector / parametric eqn

$$\pi : \underset{\sim}{r} = \underbrace{\underset{\sim}{a}}_{\text{position of vector}} + \lambda \underbrace{\underset{\sim}{b}}_{\text{direction vector}} + \mu \underbrace{\underset{\sim}{c}}_{\text{direction vector 2}} , \quad \lambda, \mu \in \mathbb{R}$$

ii. Scalar product form:

$$\pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = d \quad \text{where } d = \underset{\sim}{a} \cdot \underset{\sim}{n} \text{ is a scalar}$$

normal vector → $\underset{\sim}{b} \times \underset{\sim}{c} = \underset{\sim}{n}$

iii. cartesian equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d \Rightarrow n_1 x + n_2 y + n_3 z = d$$

converting from cartesian to vector: eg: $\pi = \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 4$

cartesian eqn is $-x + 2y - 5z = 4$

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 + 2y - 5z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ vector eqn: $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

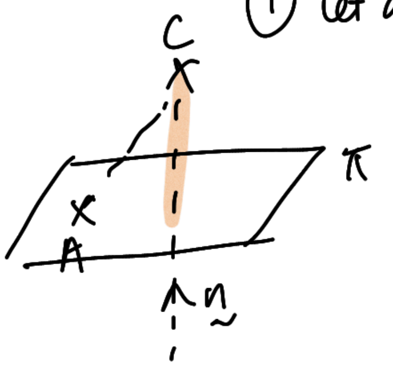
B: POINT & PLANE

i. determining if point lies on plane

① substitute point into π (For π: $\underset{\sim}{r} \cdot \underset{\sim}{n} = d$ sub position vector of pt. into $\underset{\sim}{r}$)

\# if it equals to d, then pt. lies on plane

ii. Distance from point C to a plane

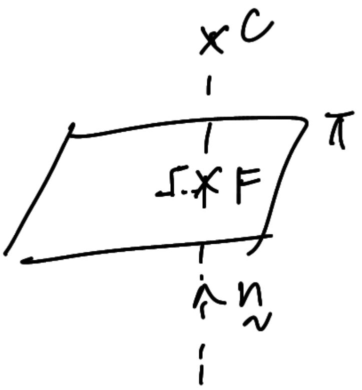


① Let a point on π be A. $\overrightarrow{OA} = (\ )$ either a given pt. on plane by qn, or just find any pt. on the plane yourself.

② use dot product

distance = $|\overrightarrow{AC} \cdot \hat{\underset{\sim}{n}}|$

iii. Finding foot of perpendicular F from pt. C to plane



① l_{CF} : $\underset{\sim}{r} = \underset{\sim}{c} + \lambda \underset{\sim}{n}$ ★

② $\overrightarrow{OF}$ is pt of intersection of l_{CF} and π

=> sub l_{CF} into π

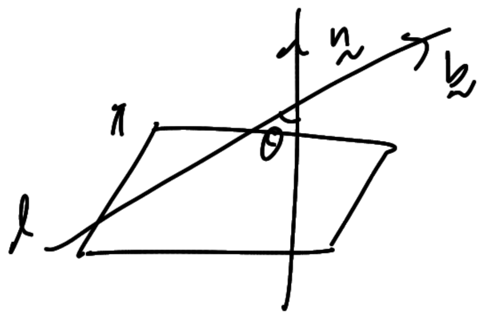
$(\underset{\sim}{c} + \lambda \underset{\sim}{n}) \cdot \underset{\sim}{n} = d$

solve for λ

③ sub λ into l_{CF} to get $\overrightarrow{OF}$

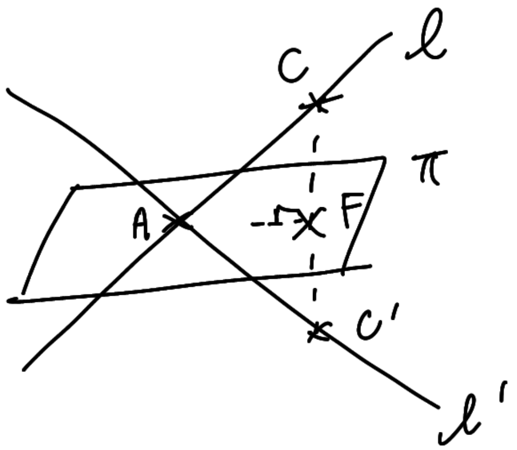
C: LINE & PLANE

i. Finding acute angle between line & plane



$$\sin\theta = \frac{|\tilde{b}\cdot\tilde{n}|}{|\tilde{b}||\tilde{n}|}$$

ii. Reflection of line about plane



① Find foot of perpendicular from a pt. on l to π.

[look @ section B iii]

② F is the mid pt of $\vec{OC}$ and $\vec{OC}'$.

so by ratio theorem,

$$\vec{OF} = \frac{1}{2}(\vec{OC} + \vec{OC}')$$

solve for $\vec{OC}$.

③ direction vector of l' is $\vec{AC}$.*

Find $\vec{AC}$.

use $\vec{OA}$ or $\vec{OC}$ as position vector.

* [NOTE: A usually found in previous parts, but if not then just find intersection l and π.]

D. PLANE & PLANE

i. Finding line of intersection of 2 planes

STEP ① Find cartesian eqns of both planes

② Put equations into GC

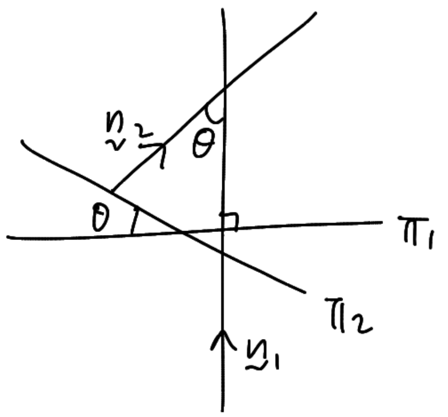
[apps → simul → 2 eqns, 3 unknowns → solve]

you'll get : eg: $x = 4 + x_3$
$x_2 = 2 - 3x_3$
$x_3 = x_3$] as solution in GC

to convert into a line : $\underset{\sim}{r} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

// ANS

ii. Acute Angle bet. 2 planes



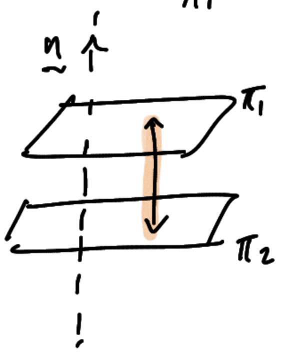
is the angle between their normals

$$\cos\theta = \frac{|\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2|}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|}$$

iii. [RARE] distance bet. 2 parallel planes

$\Pi_1 : \underset{\sim}{r} \cdot \underset{\sim}{n} = d_1$ and $\Pi_2 : \underset{\sim}{r} \cdot \underset{\sim}{n} = d_2$

☆ PARALLEL PLANES HAVE THE SAME NORMAL



$$\text{distance} = \frac{|d_1 - d_2|}{|\underset{\sim}{n}|}$$